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Congruences on $G(1, 4)$ with split universal quotient bundle

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Abstract

This work provides a complete classification of the smooth three-folds in the Grassmann variety of lines in $\mathbb{P}^4$, for which the restriction of the universal quotient bundle is a direct sum of two line bundles. For this purpose we use the geometrical interpretation of the splitting of the quotient bundle as well as the meaning of the number of the independent global sections of each of its summands.

Keywords: Grassmann varieties; Quotient bundles; Congruences

1. Introduction

As it is well known, the geometry of a subvariety in a projective space $\mathbb{P}^N$ is given by the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ restricted to the subvariety. In the same way, the geometry of a subvariety X in $G(k, N)$, the Grassmann variety of k-planes in a projective space $\mathbb{P}^n$ is given by the restriction to it of the universal quotient bundle $\mathcal{Q}$ of rank $k + 1$. It is thus of interest to have as much information as possible of this restriction, for instance about its stability.

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A first step in this direction was given in [7], where a classification of all the smooth surfaces of $G(1,3)$ for which the restriction of $\mathcal{Q}$ splits is given. This classification fits into the more general problem of studying the stability of the restriction. In fact, an important conjecture by Dolgachev and Reider states that the restriction of $\mathcal{Q}$ is semistable unless the surface is contained in a hyperplane of $\mathbb{P}^5$, the Plücker ambient space of $G(1,3)$.

In this paper we deal with the same problem for three-folds inside $G(1,4)$ (also called *congruences*). Our goal is to classify all the smooth congruences Y in $G(1,4)$ for which $\mathcal{Q}_{|Y}$ splits. The same problem for the universal subbundle of rank three was already solved in [2].

Geometrically the fact that the restriction of the universal quotient bundle splits as $\mathcal{Q}_{|Y} = \mathcal{L}_1 \oplus \mathcal{L}_2$ means that the image of Y in the Grassmannian of lines in $\mathbb{P}(H^0(\mathcal{Q}_{|Y}))$ is as a subset of the set of lines joining the points lying in two disjoint linear subvarieties of $\mathbb{P}(H^0(\mathcal{Q}_{|Y}))$ with respective dimensions $h^0(\mathcal{L}_1)-1$ and $h^0(\mathcal{L}_2)-1$. We use this fact to make the classification depending on $h^0(\mathcal{Q}_{|Y})$ which essentially amounts to making it to depend on the dimension of two linear subspaces meeting all the lines of the congruence. In this way, we see that either the congruence comes from a projection of another Grassmannian of bigger dimension or all of its lines meet a linear space of small dimension. Analyzing each of these very special properties we conclude our classification.

The structure of the paper is as follows. In a first section we give the preliminaries, a list of examples of congruences with split universal bundle (which will be eventually the complete list) and state our classification result, explaining the different non-trivial cases to study. In the second section we complete the classification by studying separately each of the different cases.

The results of this paper is part of the PhD dissertation of the second author, under the supervision of the first one.

2. Preliminaries, examples and first cases

2.1. Preliminaries

We will work over the field $\mathbb{C}$ of complex numbers. On $G(1,4)$, the Grassmann variety of lines in $\mathbb{P}^4$, we will consider the subbundle and quotient bundles appearing in the exact sequence

$$0 \to \mathcal{S}^\vee \longrightarrow V \otimes \mathcal{O}_G \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0. \tag{1}$$

Each congruence Y of $G(1,4)$ has a bidegree (a,b) where a is the number of lines of Y passing through a general point of $\mathbb{P}^4$ and b is the number of lines of Y contained in a general hyperplane H and meeting a general line of H.

A *fundamental curve* of a congruence is a curve that meets all lines of the congruence.

If $Y \subset G(1,4)$ is a congruence, the dimension of $H^0(\mathcal{Q}_{|Y})$ determines the Grassmannian in which Y fits in a natural way. More precisely, we will say that a subvariety $X \subset G(1,N)$ is *nondegenerate* if the union of all the lines of X is not contained in a hyperplane of $\mathbb{P}^N$. Then $h^0(\mathcal{Q}_{|Y}) = N+1$ means that Y is isomorphic to a nondegenerate subvariety $X \subset G(1,N)$

via a projection from $G(1, N)$ to $G(1, 4)$ induced by a linear projection from $\mathbb{P}^N$ to $\mathbb{P}^4$. We will say that $Y \subset G(1, 4)$ is *linearly normal* if $h^0(\mathcal{Q}_{|Y}) = 5$. It is immediate to observe that Y is linearly normal if and only if $h^0(G(1, 4), \mathcal{Q} \otimes \mathcal{I}_Y) = 0$, where $\mathcal{I}_Y$ is the ideal sheaf of Y in $G(1, 4)$.

2.2. Examples

Example 2.1. Let Y be a congruence in $G(1, 4)$ for which there exists a point $p \in \mathbb{P}^4$ contained in all the lines of Y. Because of dimensional reasons, it is easy to see that Y is in fact the congruence of all lines in $\mathbb{P}^4$ passing through p. Hence, $Y \cong \mathbb{P}^3$, $\mathcal{Q}_{|Y} \cong \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, the congruence Y is the zero locus of a global section of $\mathcal{S}$ and its bidegree is $(1, 0)$. Finally, we recall from [5] Lemma 4.1 that this is the only congruence of bidegree $(1, 0)$.

Example 2.2. The classification given in [5] shows that there is a unique smooth congruence of bidegree $(1, 1)$ in $G(1, 4)$. This is described as the set of lines meeting a line $L \subset \mathbb{P}^4$ and a skew plane $\Pi \subset \mathbb{P}^4$. As an abstract three-fold, Y is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and $\mathcal{Q}_{|Y}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(0, 1)$. Obviously L is a fundamental line and Y is linearly normal.

Example 2.3. Again from [5] we see that there is only one type of smooth congruence of bidegree $(2, 1)$ in $G(1, 4)$. This is described in the following way (see [1]). There is a hyperplane $H \subset \mathbb{P}^4$ containing a line L', and there exists another line $L \subset \mathbb{P}^4$ (not contained in H) with a fixed isomorphism φ from L to the pencil of planes containing L' inside of H. The congruence is given by the union, when varying $p \in L$, of the sets of lines passing through p and contained in the linear span of p and $\varphi(p)$. From this description it is clear that, as an abstract three-fold, Y is isomorphic to the blow-up $\tilde{\mathbb{P}}^3(\ell)$ of $\mathbb{P}^3$ along a line ℓ (equivalently, a hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, as described in [5]), and $\mathcal{Q}_{|Y}$ is isomorphic to $\mathcal{O}(H) \oplus \mathcal{O}(H - E)$, where H is the pullback of the hyperplane section of $\mathbb{P}^3$ and E is the exceptional divisor. It is also clear that L is a fundamental line and that Y is not linearly normal, since it comes from a nondegenerate three-fold in $G(1, 5)$.

Example 2.4. We study here one of the three types of smooth congruences of bidegree $(2, 2)$ in the classification given in [5]. As an abstract variety, Y is the projectivization $\mathbb{P}(T_{\mathbb{P}^2})$ of the tangent vector bundle of the projective plane, and hence it can be regarded as the incidence subvariety inside $\mathbb{P}^2 \times \mathbb{P}^{2*}$. It has therefore two different structures as a scroll in lines over a projective plane. Each of them yields a plane meeting all the lines of Y. Moreover, if we call these two planes Π and Π', there is an isomorphism φ between the plane Π and the set of lines in Π'. The congruence consists thus in the union of the pencils of lines passing though a point p and contained in the plane spanned by p and $\varphi(p)$. It is easy to see from this description that $\mathcal{Q}_{|Y}$ is isomorphic to the restriction to Y of $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2*}}(0, 1)$. In particular, Y is not linearly normal, and it is a projection from a nondegenerate three-fold in $G(1, 5)$.

Remark 2.5. It will be useful later on to have some information about the two other types of smooth congruences of bidegree (2, 2) different from the examples above. One of them is given as the dependency locus of three sections of $\mathcal{Q} \oplus \mathcal{S}$, and it is easy to see that it is linearly normal, since one can compute $h^1(G(1,4), \mathcal{Q} \otimes \mathcal{I}_Y) = 0$. The other one has a fundamental line, and then it is linearly normal by using Theorem 3.1 below (or a direct computation of $h^1(G(1,4), \mathcal{Q} \otimes \mathcal{I}_Y)$ using the resolution of $\mathcal{I}_Y$ given in [5]).

Example 2.6. We reproduce here an example from [3]. The vector bundle $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ induces an embedding of $\mathbb{P}^3$ in $G(1,7)$, which can be viewed in coordinates as the set of lines spanned by the rows of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 \end{pmatrix}$$

when $(x_0 : x_1 : x_2 : x_3)$ varies in $\mathbb{P}^3$. We consider now the linear projection from $\mathbb{P}^7$ to $\mathbb{P}^4$ defined by

$$(z_0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6 : z_7) \mapsto (z_0 : z_1 + z_4 : z_2 + z_5 : z_3 + z_6 : z_7).$$

This projection induces a projection from $G(1,7)$ to $G(1,4)$ and the image of $\mathbb{P}^3$ in $G(1,4)$ is given by the lines spanned by the rows of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & 0 \\ 0 & x_0 & x_1 & x_2 & x_3 \end{pmatrix}.$$

Since the minors of this matrix form a basis of the set of homogeneous polynomials of degree two, the composition of $\mathbb{P}^3 \longrightarrow G(1,4)$ with the Plücker's embedding of $G(1,4)$ in $\mathbb{P}^9$ defines precisely the double Veronese embedding of $\mathbb{P}^3$. In particular, the first morphism is an embedding. Therefore, its image is a smooth congruence in $G(1,4)$, of bidegree (4, 2), which is not linearly normal. In fact, in [3] it is proved that this is the only smooth congruence of $G(1,4)$ that is projected from a nondegenerate three-fold in $G(1,7)$. It is also obvious that $\mathcal{Q}_{|Y}$ is $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$.

Example 2.7. We construct now an example of a smooth congruence of bidegree (3, 2) whose existence remained unknown in [5]. This example is in fact analogous to Examples 2.3 and 2.6. We consider the blow-up $\tilde{\mathbb{P}}^3(p)$ of $\mathbb{P}^3$ in a point p and the vector bundle $\mathcal{O}_{\tilde{\mathbb{P}}^3(p)}(H - E) \oplus \mathcal{O}_{\tilde{\mathbb{P}}^3(p)}(H)$, where H is the pullback of the hyperplane section of $\mathbb{P}^3$ and E is the exceptional divisor. This gives an embedding of $\tilde{\mathbb{P}}^3(p)$ into $G(1,6)$, which in coordinates, and assuming that $p = (1 : 0 : 0 : 0)$, can be described as the set of lines spanned by the rows of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 & x_3 \end{pmatrix}$$

when $(x_0 : x_1 : x_2 : x_3)$ varies in $\mathbb{P}^3$ (we make an abuse of notation, and for instance the entries in the first row of the above matrix should be in fact sections of $\mathcal{O}_{\tilde{\mathbb{P}}^3}(p)(H-E)$). We consider now the linear projection from $\mathbb{P}^6$ to $\mathbb{P}^4$ given by

$$(z_0 : z_1 : z_2 : z_3 : z_4 : z_5 : z_6) \mapsto (z_3 : z_4 : z_5 + z_0 : z_6 + z_1 : z_2)$$

and we have, as in the previous example, that the image Y of $\tilde{\mathbb{P}}^3(p)$ in $G(1,4)$ by this projection is given by the rows of the matrix

$$\begin{pmatrix} 0 & 0 & x_1 & x_2 & x_3 \\ x_0 & x_1 & x_2 & x_3 & 0 \end{pmatrix}$$

The minors of this matrix still form a basis of the linear system of quadrics passing through p. Therefore, this yields an example of a smooth congruence of $G(1,4)$ that comes from a nondegenerate three-fold in $G(1,6)$.

2.3. Statement of the main theorem

The main result that we will prove in this paper is the following:

Theorem 2.8. *If Y is a smooth congruence in $G(1,4)$ such that the restriction of the universal quotient bundle $\mathcal{Q}$ splits, the pair $(Y, \mathcal{Q}_{|Y})$ is one of the following*:

(i) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$, *as in* Example 2.1.
(ii) $(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,1))$, *as in* Example 2.2.
(iii) $(\tilde{\mathbb{P}}^3(\ell), \mathcal{O}_{\tilde{\mathbb{P}}^3(\ell)}(H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}^3(\ell)}(H-E))$, *as in* Example 2.3.
(iv) $(\mathbb{P}(\mathrm{T}_{\mathbb{P}^2}), \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}}(1,0)_{|Y} \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^{2*}}(0,1)_{|Y})$, *as in* Example 2.4.
(v) $(\tilde{\mathbb{P}}^3(p), \mathcal{O}_{\tilde{\mathbb{P}}^3(p)}(H) \oplus \mathcal{O}_{\tilde{\mathbb{P}}^3(p)}(H-E))$, *as in* Example 2.7.
(vi) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$, *as in* Example 2.6.

In order to prove this theorem, we assume that $\mathcal{Q}_{|Y}$ splits as $\mathcal{L}_1 \oplus \mathcal{L}_2$, and suppose, without loss of generality, that $h^0(\mathcal{L}_1) \le h^0(\mathcal{L}_2)$. Since $\mathcal{Q}_{|Y}$ is globally generated, the same holds for $\mathcal{L}_1$ and $\mathcal{L}_2$, and in particular they define a map from Y to linear spaces of respective dimensions $h^0(\mathcal{L}_1) - 1$ and $h^0(\mathcal{L}_2) - 1$ that meet all the lines of Y. We distinguish the following cases:

(a) $h^0(\mathcal{L}_1) = 1$: Then Y is contained in the set of lines on $\mathbb{P}^4$ passing through a point. Hence Y is as in Example 2.1, and it is case (i) in Theorem 2.8.
(b) $h^0(\mathcal{L}_1) = 2$: Then there is a line L meeting all the lines of Y. If $h^0(\mathcal{L}_2) = 3$, then all the lines of Y also meet a plane Π, so that Y is necessarily the congruence of Example 2.2, i.e. case (ii) in Theorem 2.8.
(c) $h^0(\mathcal{L}_1) \ge 3$: Then $h^0(\mathcal{Q}_{|Y}) \ge 6$. This implies that Y is a projection from at least $G(1,5)$. It is proved in [3] that if $h^0(\mathcal{Q}_{|Y}) \ge 8$ then Y is necessarily the Veronese variety of Example 2.6, i.e. case (vi) in Theorem 2.8.

The above discussion implies that we are left with the subcase $h^0(\mathcal{L}_1) = 2$, $h^0(\mathcal{L}_2) \geq 4$ in case (b) and the subcases $h^0(\mathcal{L}_1) = h^0(\mathcal{L}_2) = 3$; $h^0(\mathcal{L}_1) = 3$, $h^0(\mathcal{L}_2) = 4$ in case (c). We will devote the next section to the study of these subcases, which will complete the proof of Theorem 2.8.

3. Proof of the main theorem

3.1. The case $h^0(\mathcal{L}_1) = 2, h^0(\mathcal{L}_2) \geq 4$

For such a congruence we observe that, besides the property of possessing a fundamental line L, Y also satisfies $h^0(\mathcal{Q}_{|Y}) \geq 6$, hence Y is not linearly normal. In [4] there is a description of all the congruences of $G(1, 4)$ with a fundamental line, so the natural way of studying this case would be to compute for each of them the dimension of $H^0(\mathcal{Q}_{|Y})$, and check that only the one of Example 2.3 is not linearly normal. This was the method used in [8], which required a lot of tedious computations. Here we will use instead an alternative geometric approach suggested to us by José Carlos Sierra.

Theorem 3.1. *Let Y be a smooth congruence of $G(1, 4)$ with fundamental line L that is not the congruence of* Example 2.3. *Then Y is linearly normal.*

Proof. Assume that we have a nondegenerate subvariety $\overline{Y} \subset G(1, 5)$ such that all of its lines meet a fixed line $L \subset \mathbb{P}^5$ and that is smoothly projectable to $G(1, 4)$. Through each point p of L there passes a two-dimensional family of lines of $\overline{Y}$, whose union is a cone with vertex p. The secant variety of this cone cannot be the whole $\mathbb{P}^5$, since otherwise projecting from any point of $\mathbb{P}^5$ two different lines of $\overline{Y}$ would go to the same line in $\mathbb{P}^4$, contrary to our projectability hypotheses. Hence either the cone is a $\mathbb{P}^3$ or a hypersurface contained in a hyperplane of $\mathbb{P}^5$. In the second case, the hyperplane should be the same for all the points p of L, since otherwise the union of all the hyperplanes will be $\mathbb{P}^5$, contradicting again the hypotheses that $\overline{Y}$ is smoothly projectable to $G(1, 4)$; but this means that $\overline{Y}$ is degenerate, which is absurd. Therefore, we have that the set of lines of $\overline{Y}$ through each point p of L is the set of lines passing through p and contained in a three-dimensional linear space A_p containing p.

Looking at the description of congruences with a fundamental line given in [4], we see that they are either of bidegree $(b, b-1)$ or (b, b) and that b is the degree of the cone of all the lines passing through a general point p of the fundamental line. We have seen that in our case this cone is A_p, or rather its projection to $\mathbb{P}^4$, which has degree one. Hence the only possibilities are then a congruence of bidegree $(1, 1)$ (which is the one of Example 2.2 and is linearly normal) and a congruence of bidegree $(2, 1)$, which is Example 2.3. This completes the proof. □

3.2. The case $h^0(\mathcal{L}_1) = h^0(\mathcal{L}_2) = 3$

To classify the congruences in this case, we prove the following theorem.

Theorem 3.2. *Let $Y \subset G(1,4)$ be a congruence that is projected from a nondegenerate variety $\overline{Y} \subset G(1,5)$ such that all lines of $\overline{Y}$ meet two disjoint planes Π_1 and Π_2 of $\mathbb{P}^5$. Then, Y is the congruence of* Example 2.4.

Proof. The set of lines in $\mathbb{P}^5$ meeting Π_1 and Π_2 can be identified with $\mathbb{P}^2 \times \mathbb{P}^2$ embedded in $G(1,5)$ by the vector bundle $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0) \oplus \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)$. Hence $\overline{Y}$ is a divisor of $\mathbb{P}^2 \times \mathbb{P}^2$. The Picard group of $\mathbb{P}^2 \times \mathbb{P}^2$ is generated by the pullbacks h_1 and h_2 of the hyperplane sections of each of its two factors. Therefore the class of $\overline{Y}$ in the Chow ring of $\mathbb{P}^2 \times \mathbb{P}^2$ can be written as

$$[\overline{Y}] = \alpha h_1 + \beta h_2,$$

for some $\alpha, \beta \in \mathbb{Z}$. It is easy to check that the bidegree of the projected Y is

$$(a,b) = (\alpha+\beta, \alpha+\beta).$$

The idea now is to see that the fact that $\overline{Y}$ is smoothly projectable to $G(1,4)$ gives a very strong numerical conditions on α and β.

Let $p \in \mathbb{P}^5$ be the center of the projection from $\mathbb{P}^5$ to $\mathbb{P}^4$ that induces the isomorphism from $\overline{Y}$ to Y. Since the projection can be taken to be general, we can assume that p is not in any Π_i (we could also use Theorem 3.1, since such a special projection would produce a congruence with a fundamental line). Define

$$p_1 = \langle p, \Pi_1 \rangle \cap \Pi_2 \quad \text{and} \quad p_2 = \langle p, \Pi_2 \rangle \cap \Pi_1.$$

The points p, p_1, p_2 are in the line $\langle p, \Pi_1 \rangle \cap \langle p, \Pi_2 \rangle$ whose image under the projection from p gives the intersection point of the planes that are the image of Π_1 and Π_2.

Obviously, for any line L contained in Π_2 (resp. Π_1), all the lines of the pencil of lines passing through p_1 (resp. p_2) and contained in the span of p_1 (resp. p_2) and L are projected to the same line of $\mathbb{P}^4$ (in fact, it is not difficult to see that these are the only possible contractions for the set of lines meeting Π_1 and Π_2). It thus follows that any of these pencils can only contain at most one line of $\overline{Y}$. Since the class in the Chow ring of $\mathbb{P}^2 \times \mathbb{P}^2$ of any of these pencils is $h_1^2 h_2$ (resp. $h_1 h_2^2$), it follows that the intersection numbers $[\overline{Y}]h_1^2 h_2$ and $[\overline{Y}]h_1 h_2^2$ are at most one, i.e. $\alpha, \beta \leq 1$.

Therefore, the only possible bidegrees for Y are $(1,1)$ and $(2,2)$. The only smooth congruence of bidegree $(1,1)$ in $G(1,4)$ is the one of Example 2.2, which is linearly normal, and hence it does not appear in this case. As for bidegree $(2,2)$, the only possible congruences are the one of Example 2.4 and the ones of Remark 2.5. Since the latter are linearly normal, we conclude the proof. □

Remark 3.3. A different way of ruling out the case of bidegree $(1,1)$ in the last proof would be to observe that $\alpha = 0$ (resp. $\beta = 0$) implies that through a general point of Π_1 (resp. Π_2) there do not pass lines of $\overline{Y}$, which implies that there exists a fundamental curve in Π_1 (resp. Π_2). Then it is enough to use the classification given in [4]. This different approach can be found in [8].

3.3. The case $h^0(\mathcal{L}_1) = 3, h^0(\mathcal{L}_2) = 4$

To complete the study of this case (and hence the proof of Theorem 2.8) we will prove the following theorem.

Theorem 3.4. *The only smooth congruence* Y *in* $G(1,4)$ *that comes from a nondegenerate three-fold* $\overline{Y} \subset G(1,6)$ *and for which the restricted universal vector bundle* $\mathcal{Q}_{|Y}$ *splits with* $h^0(\mathcal{L}_1) = 3$ and $h^0(\mathcal{L}_2) = 4$ *is the congruence of* Example 2.7.

Notation 3.5. Throughout the proof, which we will divide into several steps along this subsection, we will write $\Pi_1 = \mathbb{P}(H^0(\mathcal{L}_1))$ (of dimension two), and $\Lambda_2 = \mathbb{P}(H^0(\mathcal{L}_2))$ (of dimension three), the two disjoint linear subvarieties of $\mathbb{P}^6$ that meet all lines of $\overline{Y}$.

Lemma 3.6. *If* $\overline{Y}$ *is a variety of* $G(1,6)$ *that can be isomorphically projected to* $G(1,4)$ *in the above conditions, then through any point of* Π_1 *there passes some line of* $\overline{Y}$*, while the dimension of the set of points of* Λ_2 *through which there passes some line of* $\overline{Y}$ *is at least two.*

Proof. If the statement were not true, then the (image of the) set of lines of Π_1 or Λ_2 meeting some line of $\overline{Y}$ would be a fundamental curve for Y. But the classification of smooth congruences in $G(1,4)$ with a fundamental curve given in [4] does not contain any congruence that is projected from a nondegenerate three-fold in $G(1,6)$. □

As in the previous subsection, we can view $\overline{Y}$ as a subvariety of $\mathbb{P}^2 \times \mathbb{P}^3$ embedded in $G(1,6)$ by $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,0) \otimes \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^3}(0,1)$. Then the class of $\overline{Y}$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is:

$$[Y] = \alpha h_1^2 + \beta h_1 h_2 + \gamma h_2^2,$$

where h_1 denotes the pullback of the class of the hyperplane section of $\mathbb{P}^2$, h_2 is the pullback of the class of the hyperplane section of $\mathbb{P}^3$, and α, β, γ are nonnegative integers. An easy calculation shows that the bidegree of the projected $Y \subset G(1,4)$ is

$$(a, b) = (\alpha + \beta + \gamma, \beta + \gamma).$$

As in the previous subsection, the idea is to bound these numbers and to use this to calculate the possible bidegrees of Y.

Lemma 3.7. *In the above conditions,* $\beta = 0$ *or* $\beta = 1$.

Proof. Let $L \subset \mathbb{P}^6$ be the center of the linear projection from $\mathbb{P}^6$ to $\mathbb{P}^4$ that induces the projection from $G(1,6)$ to $G(1,4)$ sending $\overline{Y}$ to Y. Since the projection is general, we can assume that L does not meet neither Π_1 nor Λ_2. Consider the lines

$$L_1 = \langle L, \Lambda_2 \rangle \cap \Pi_1 \quad \text{and} \quad L_2 = \langle L, \Pi_1 \rangle \cap \Lambda_2.$$

(which are the lines of Λ_2 and Π_1 whose image by the projection is precisely the intersection of the images of Π_1 and Λ_2). Consider in $\mathbb{P}^2 \times \mathbb{P}^3$ the subset Q consisting of the lines of

$\mathbb{P}^6$ meeting the lines L_1 and L_2. Its class in the Chow ring of $\mathbb{P}^2 \times \mathbb{P}^3$, is $h_1 h_2^2$, and by construction we have that all the elements Q have the same image in $G(1,4)$ under the linear projection from L. Since $\overline{Y}$ can be projected isomorphically from L, it follows that the intersection product of the classes of $\overline{Y}$ and Q is at most one. This immediately yields $\beta \leq 1$, as desired. □

Take now a general $p_1 \in \Pi_1$. By Lemma 3.6, there is a one dimensional family of lines of $\overline{Y}$ passing through p_1. The intersection of all these lines with A_2 will yield a curve C_{p_1} in A_2.

Lemma 3.8. *The general curve C_{p_1} is a line.*

Proof. The cone with vertex p_1 over a secant line to C_{p_1} is obviously a plane containing two lines of $\overline{Y}$. If we assume for contradiction that C_{p_1} is not a line, then the secant variety of C_{p_1} has dimension at least two (in fact it would be three unless C_{p_1} were a plane curve). This gives a two-dimensional family of planes containing two lines of $\overline{Y}$. When moving the point p_1 in Π_1, we get a four-dimensional family of such planes (observe that any of these planes correspond to a unique point $p_1 \in \Pi_1$). Since $\overline{Y}$ is projectable to $G(1,4)$, the union of the set of planes containing two lines of $\overline{Y}$ has dimension at most four. But since this union contains a four-dimensional family of planes, it must be either a linear space (which is absurd, since $\overline{Y}$ is nondegenerate) or it is made out of a one-dimensional family of linear spaces of dimension three. The latter case is also impossible, since this would mean for instance that any of these planes must meet a three-dimensional family of other planes along a line, which is absurd. □

Notation 3.9. From now on we will write r_{p_1} instead of C_{p_1}. We will also write V for the family of lines $r_{p_1} \subset A_2$ when p_1 varies in Π_1.

Lemma 3.10. *The family V has dimension two.*

Proof. Since V is dominated by Π_1, it has dimension at most two. It cannot consist of just one line, since this would be a fundamental line. Assume for contradiction that V has dimension one. Then, a general line $r \in V$ comes from the points of a curve $C_r \subset \Pi_1$. Reasoning as in the proof of the previous lemma, we observe that the cone with vertex r over the secant variety C_r is made out of planes containing two lines of $\overline{Y}$. This implies that C_r must be a line.

Observe that then for any $r \in V$ we have that the set of lines meeting r and C_r is contained in $\overline{Y}$, and this is a quadric surface after the Plücker embedding. Hence $\overline{Y}$ is given as a union of quadric surfaces.

Observe also that through a general point of Π_1 there passes only one line C_r (because a general point $p_1 \in \Pi_1$ determines a unique line r_{p_1}). Hence the set of lines C_r is a pencil in Π_1. In particular, $\overline{Y}$ is a union of quadrics parameterized by a rational curve.

We can use now the classification done in [6] Theorem 2.2 of smooth congruences $Y \subset G(1,4)$ with a quadric bundle structure, and we check that $h^0(\mathcal{Q}_{|Y}) \leq 6$ for all of them for which the base curve is rational. This is a contradiction, which completes the proof. □

Lemma 3.11. Through a general point of A_2 there passes a unique line of V.

Proof. Assume that through a general point $p_2 \in A_2$ there pass at least two lines r_{p_1} and $r_{p'_1}$. Then the plane spanned by $p_1,\ p'_1,\ p_2$ contains the lines $p_1p_2,\ p'_1p_2$ of $\overline{Y}$. Since through a general p_2 there pass finitely many of these planes, we obtain a three-dimensional family of planes whose union has necessarily dimension five. This contradicts the projectability of $\overline{Y}$. □

We could finish now the proof of Theorem 3.4 by the classification given in [9] of the surfaces of $G(1, 3)$ such that through a general point of $\mathbb{P}^3$ there passes a unique line of the surface. We however prefer to finish the proof in a more geometric way.

Lemma 3.12. *With the above notation,* $\alpha = 1$.

Proof. Since α is the intersection number $[\overline{Y}] \cdot h_2^3$, then geometrically it represents the number of lines of $\overline{Y}$ passing through a general point of A_2. So let $p_2 \in A_2$ be a general point. We thus know by Lemma 3.11 that there is a unique line r_{p_1} passing through p_2. Since then p_1p_2 is in $\overline{Y}$, it follows that $\alpha > 0$. It could happen however that there is another $p'_1 \in \Pi_1$ such that $r_{p'_1} = r_{p_1}$. But the same proof as for Lemma 3.11 excludes this possibility. □

Lemma 3.13. *With the above notation,* $\gamma = 1$.

Proof. As before, $\gamma = [\overline{Y}] \cdot h_1^2 \cdot h_2$ is the number of lines of $\overline{Y}$ passing through a general point $p_1 \in \Pi_1$ and meeting a general plane $H \subset A_2$. Given $p_1 \in \Pi_1$, let r_{p_1} be its corresponding line in A_2. Taking a plane H not containing r_{p_1}, and writing $q = H \cap r_{p_1}$, then $\langle p_1, q\rangle$ is the only one line of Y passing through p_1 and meeting H. Thus, $\gamma = 1$. □

The proof of Theorem 3.4 follows now readily. From the above lemmas, the class of $\overline{Y}$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is $h_1^2 + \beta h_1h_2 + h_2^2$, with $\beta \in \{0, 1\}$ and hence the possible bidegrees of the projected Y are $(a, b) = (2, 1)$ and $(a, b) = (3, 2)$. We have seen that the only smooth congruence of bidegree $(2, 1)$ is the one of Example 2.3, which is not a projection from a nondegenerate three-fold in $G(1, 6)$. And for bidegree $(3, 2)$, the classification in [5] gives only the case of Example 2.7 and another one with a fundamental line (which is linearly normal by Theorem 3.1).

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References

[1] A. Alzati, 3-scroll immersions in $G(1, 4)$, Ann. Univ. Ferrara 32 (1986) 45–54.
[2] E. Arrondo, The universal rank-$(n - 1)$ bundles on $G(1, n)$ restricted to subvarieties, Collect. Math. 49 (2/3) (1998) 173–183, Dedicado a la memoria de Fernando Serrano.

[3] E. Arrondo, Projections on Grassmannians of lines and characterization of Veronese varieties, J. Algebr. Geom. 8 (50) (1999) 85–101.
[4] E. Arrondo, M. Bertolini, C. Turrini, Classification of Smooth Congruences with a Fundamental Curve, vol. 166, Marcel Dekker, 1994, pp. 43–56.
[5] E. Arrondo, M. Bertolini, C. Turrini, Congruences of small degree in $G(1,4)$, Comm. Algebra 26 (10) (1998) 3249–3266.
[6] E. Arrondo, M. Bertolini, C. Turrini, Quadric bundle congruences in $G(1,n)$, Forum Math. 12 (2000) 649–666.
[7] E. Arrondo, I. Sols, On congruences of lines in the projective space, Mém. Soc. Math. France N.S. 50 (1992) 96.
[8] B. Graña, Escisión de fibrados en $G(1,4)$ y sus variedades, Ph.D. Thesis, Universidad Complutense de Madrid, January, 2003.
[9] Z. Ran, Surfaces of order 1 in Grassmannians, J. Reine Angew. Math. 368 (1986) 119–126.